University College,
London 2nd Jany. 1849.

Sir, The short treatise inclosed herewith is a very rare one – by Dr. Waring formerly Lucasian Professor of Mathematics at Cambridge. There is only one copy known to exist in London, which was formerly in the possession of Francis Baily Esqre the Mathematician and Astronomer, and now in the possession of my Colleague Professor De Morgan.

At the desire of the latter Gentleman I have procured the present copy from the library of my own College (Queens' College, Cambridge) to

present to the Library of the British Museum.
I have to beg the favour of an acknowledgement of the receipt of it, which I may forward to the Society of Queens' College, Cambridge.

I am Sir, Yours Truly

Richd Potter A.M.

Professor of Natural Philosophy and Astronomy in University College London.

To Sir Henry Ellis
Principal Librarian
of the British Museu

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ON THE PRINCIPLES OF TRANSLATING ALGEBRAIC QUANTITIES INTO PROBABLE RELATIONS AND ANNUITIES, &c.

By E. WARING, M.D.

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ON THE PRINCIPLES OF TRANSLATING ALGEBRAIC QUANTITIES, &c.

LEMMA.

1. LET $\frac{A+Br+Cr^2+Dr^3+\ldots+Hr^{l-1}+\ldots+r^{z-2}}{(1-r)^z}$ $=1+2^{z-1}\times r+3^{z-1}\times r^2+4^{z-1}\times r^3+5^{z-1}\times r^4$ $+6^{z-1}\times r^5+$ &c. in infinitum; then will $A=1$, $B=2^{z-1}-z$, $C=3^{z-1}-z\times 2^{z-1}+z.\frac{z-1}{2}$, $D=4^{z-1}-z\times 3^{z-1}+z.\frac{z-1}{2}\times 2^{z-1}-z.\frac{z-1}{2}.\frac{z-2}{3}$, ..., $H=l^{z-1}-z\times(l-1)^{z-1}+z.\frac{z-1}{2}\times(l-2)^{z-1}-z.\frac{z-1}{2}.\frac{z-2}{3}\times(l-3)^{z-1}+z.\frac{z-1}{2}.\frac{z-2}{3}.\frac{z-3}{4}\times(l-4)^{z-1}-$ &c.

The coefficient of the laſt term r^{z-2} will be 1, and all the ſubſequent coefficients become $=0$, as is manifeſt from a theor. publiſhed in the Phil. Tranſ. viz. $l^{z-1}-z(l-1)^{z-1}+z.\frac{z-1}{2}(l-2)^{z-1}$ — &c. will be $=1$, when $l=z-1$; but $=0$, when l is greater than $z-1$.

The ſame principles may be applied when z

is either negative or fractional, in which caſe the ſeries $A + Br + Cr^2 + Dr^3 +$ &c. proceeds in infinitum.

If for r in the abovementioned quantities be ſubſtituted $-r$, then the terms reſulting will be alternately + and —.

From the lemma and the principles given in the Meditat. Algebr. which were ſent in ſome papers to the Royal Society in 1757, may be found the ſum of every (s) terms of the ſeries $1 + 2^{z-1} \times r + 3^{z-1} \times r^2 +$ &c.

2. To find the ſum of the ſeries $1 + 2^{z-1} r + 3^{z-1} \times r^2 + 4^{z-1} \times r^3 + \ldots + (k-1)^{z-1} \times r^{k-2}$ conſiſting of $k-1$ terms.

The ſum will be equal to a fraction, of which the denominator is (as in the preceding lemma) $(1-r)^z$, and the numerator $A + Br + Cr^2 + Dr^3 + \ldots + Hr^{l-1} + \ldots + r^{z-2} - k^{z-1} \times r^{k-1} - ((k+1)^{z-1} - z \times k^{z-1}) \times r^k - ((k+2)^{z-1} - z \times (k+1)^{z-1} + z \cdot \frac{z-1}{2} \times k^{z-1}) \times r^{k+1} - ((k+3)^{z-1} - z \times (k+2)^{z-1} + z \cdot \frac{z-1}{2} \times (k+1)^{z-1} - z \cdot \frac{z-1}{2} \cdot \frac{z-2}{3} \times k^{z-1}) \times r^{k+2} - ((k+4)^{z-1} - z \times (k+3)^{z-1} + z \cdot \frac{z-1}{2} \times (k+2)^{z-1} - z \cdot \frac{z-1}{2} \cdot \frac{z-2}{3} \times (k+1)^{z-1} + z \cdot \frac{z-1}{2} \cdot \frac{z-2}{3} \cdot \frac{z-3}{4} \times k^{z-1}$

$\times k^{z-1})\times r^{k+3}-$ &c. to z terms; all the remaining terms, by the Meditat. Analyticæ, become $=0$: the coefficients $A, B, C, D, \ldots H$ to $z-1$ terms denote the ſame quantities as in the preceding lemma; and the law of the z terms may eaſily be ſeen from inſpection, or collected from the law of the terms in the preceding lemma.

3. This lemma may eaſily be deduced from the preceding by the following propoſition.

Let $d1$, $d2\times r$, $d3\times r^2$, $d4\times r^3$, &c. be the ſucceſſive terms of the diviſor of any fraction; and $q1$, $q2\times r$, $q3\times r^2$, $q4\times r^3$, &c. the ſucceſſive terms of the quotient; and $R.b\times r^b$, $R\overline{b'+1}\times r^{b+1}$, $R.\overline{b''+2}\times r^{b+2}$, $R.\overline{b'''+3}\times r^{b+3}$, &c. the ſucceſſive terms of the remainder, of which the diſtance from the firſt is $b-1$; that is, let the diviſor be $d1+d2\times r+d3\times r^2+d4\times r^3+$ &c.; the quotient $q1+q2\times r+q3\times r^2+q4\times r^3+$ &c.; and the remainder (whoſe diſtance from the firſt is $b-1$) $Rb\times r^b+R.\overline{b'+1}\times r^{b+1}+R.\overline{b''+2}\times r^{b+2}+R.\overline{b'''+3}\times r^{b+3}+$ &c.; then will $Rb=d1\times q.\overline{b+1}$, $R.\overline{b'+1}=d1\times q.\overline{b+2}+d2\times q.\overline{b+1}$, $R.\overline{b+2}=d1\times q.\overline{b+3}+d2\times q.\overline{b+2}+d3\times q.\overline{b+1}$, $R.\overline{b+3}=d1\times q.\overline{b+4}+d2\times q.\overline{b+3}+d3\times q.\overline{b+2}+d4\times q.\overline{b+1}$, $R.\overline{b+4}=d1\times q.\overline{b+5}+d2\times q.\overline{b+4}+d3\times q.\overline{b+3}+d4\times q.\overline{b+2}+d5\times q.\overline{b+1}$, &c.

Hence the terms of the remainders may eaſily be deduced from the terms of the diviſor and quotient.

Cor. If the terms of the quotient contained between two different remainders be required, ſubſtract the laſt remainder from the firſt, and the difference divided by the diviſor will give the terms of the quotient required.

Cor. Hence the ſum of any ſeries $(a+b+c+d+\&c.)+(a\times 2^{z-1}+b\times 2^{z-2}+c\times 2^{z-3}+d\times 2^{z-4}+\&c.)r+(a\times 3^{z-1}+b\times 3^{z-2}+c\times 3^{z-3}+\&c.)r^2+(a\times 4^{z-1}+b\times 4^{z-2}+c\times 4^{z-3}+\&c.)r3+\&c.$ may be deduced; for the ſum of each of the terms multiplied into a is found by the lemma; and in like manner may be found the ſum of each of the terms multiplied into b, into c, &c.; and conſequently by adding them together the ſum of the whole ſeries.

Of tranſlating propoſitions expreſſing the relation between algebraical quantities to others expreſſing probable relations.

1. Certainty being contained in the whole number of events poſſible, may be expreſſed by any given quantity or number; e. g. N or 1.

2. The probability or chance $\left(\frac{a}{N}\text{ or }a\right)$ will always be leſs than the quantity or number abovementioned N or 1; and conſequently its reciprocal greater than $\frac{1}{N}$ or 1.

3. Let a and b be the chances of two events A and B independent of each other reſpectively

ſucceeding, and certainty N; then will $\frac{a\times b}{N^2}$ be the chance of both the events ſucceeding in the two trials; and $\frac{\overline{N-b}\times a}{N^2}$ be the chance of B's failing in one and A's ſucceeding in the other; and $\frac{b\times\overline{N-a}}{N^2}$ the chances of B's ſucceeding and A's failing; and $\frac{\overline{N-a}\,.\,\overline{N-b}}{N^2}$ the chance of A's and B's both failing.

For, if $\frac{a}{N}$ be the chance of A's ſucceeding, $\frac{N-a}{N}$ will be the chance of its failing in the ſame trial.

4. Let a, b, c, d, &c.; α, β, γ, δ, &c. be the chances of any independent events A, B, C, D, &c.; A′, B′, Γ, Δ, &c. ſucceeding to certainty N; then will the chance of all the events ſucceeding be $\frac{a}{N}\times\frac{b}{N}\times\frac{c}{N}\times\frac{d}{N}\times$ &c. $\times\frac{\alpha}{N}\times\frac{\beta}{N}\times\frac{\gamma}{N}\times\frac{\delta}{N}\times$ &c. The chance of the events A, B, C, D, &c. ſucceeding, and the events A′, B′, Γ, Δ, &c. failing will be $\frac{a}{N}\times\frac{b}{N}\times\frac{c}{N}\times\frac{d}{N}\times$ &c. $\times\frac{N-\alpha}{N}\times\frac{N-\beta}{N}\times\frac{N-\gamma}{N}\times$ &c.

5. The chance of the event of A ſucceeding n times together, and of B ſucceeding m times,

and C happening r times, &c.; and of A' failing n' times together, and B' failing m' times, and Γ failing r' times, &c. will be $\frac{a^n}{N^n} \times \frac{b^m}{N^m} \times \frac{c^r}{N^r} \times$ &c. $\times \left(\frac{N-\alpha}{N}\right)^{n'} \times \left(\frac{N-\beta}{N}\right)^{m'} \times \left(\frac{N-\gamma}{N}\right)^{r'} \times$ &c.

Theſe events are ſuppoſed to be independent of each other, or to happen in a given order.

6. If the order of two dependent events, as A and B, be not fixed; i. e. the event A may happen firſt and B ſecond, or B may happen firſt and A ſecond; then will the chance of the events A and B happening in the two trials be $2 \times \frac{a}{N} \times \frac{b}{N}$.

In the ſame manner the chance of A's happening n times, B's happening m times, C's happening r times, and A''s failing n' times, B' failing m' times, Γ failing r' times, &c. in $n+m+r+n'+m'+r'+$ &c. $=l$ trials in every order poſſible, will be $l \times \frac{l-1}{2} \times \frac{l-2}{3} \cdots \frac{l-m+1}{m} \times \frac{l-m}{1} \times \frac{l-m-1}{2} \times \frac{l-m-2}{3} \cdots \frac{l-m-r+1}{r} \times \frac{l-m-r}{1} \times \frac{l-m-r-1}{2} \times \frac{l-m-r-2}{3} \cdots \frac{l-m-r-n'+1}{n'} \times \frac{l-m-r-n'}{1} \times \frac{l-m-r-n'-1}{2} \times \frac{l-m-r-n'-2}{3} \cdots \frac{l-m-r-n'-m'+1}{m'} \times$

$l-m$

$$\frac{l-m-r-n'-m'}{1} \times \frac{l-m-n-n'-m'-1}{2} \times \frac{l-m-r-n'-m'-2}{3} \ldots \frac{l-m-r-n'-m'-r'+1}{r'}$$

$$\times \&c. (H) \times \frac{a^n}{N^n} \times \frac{b^m}{N^m} \times \frac{c^r}{N^r} \times \left(\frac{N-\alpha}{N}\right)^{n'} \times \left(\frac{N-\beta}{N}\right)^{m'}$$

$$\times \left(\frac{N-\gamma}{N}\right)^{r'} \times \&c.$$

7. From theſe principles any quantities may be tranſlated into the probabilities of events. e.g. 1. Any power $\frac{a^n}{N^n} \times H$ may be tranſlated into the probability of an event's happening *n* times (a whole number) of which the chance of its happening each time is $\frac{a}{N}$, multiplied into *H*; in this caſe *a* is ſuppoſed leſs than *N*: but if *a* be greater than *N*, it muſt be conſtrued the reciprocal of the abovementioned probability. 2. $\overline{\frac{a}{N}}\Big|^{\frac{1}{m}}$ may be conſtrued the probability or chance of an event's ſucceeding in one time, of which the chance of its happening (*m*) times together is $\frac{a}{N}$. 3. $\overline{\frac{a}{N}}\Big|^{\frac{n}{m}}$ denotes the chance of an event's happening (*n*) times together, of which the chance of its happening (*m*) times together is $\frac{a}{N}$.

4. $H \times \frac{a^n}{N^n} \times \frac{b^m}{N'^m}$ may be conſtrued the chance of the event's (A) happening n times together, and the event's (B) happening (m) times together (of which the reſpective chances of happening each time are $\frac{a}{N}$ and $\frac{b}{N'}$) multiplied into H; if the events are ſuppoſed independent of each other, or to happen in a given order.

5. The quantity $\left(H + K a^{\frac{n}{m}} \times b^p + L \times a^{\frac{b}{k}}\right)^{\frac{r}{s}}$ may denote the chance of an event (S) happening (r) times together, of which the chance of its happening (s) times together is $H + K a^{\frac{n}{m}} \times b^p + l \times a^{\frac{b}{k}}$; which quantity may be tranſlated into probable relations from the principles before given.

In the ſame manner may all algebraical and exponential quantities, and their naſcent or finite increments be tranſlated into ſimilar quantities. If a be greater than N, or b than N'; then for the probabilities or chances of happening or failing always read their reciprocals.

8. Any algebraical, &c. equations may be tranſlated from theſe principles into equations expreſſing the relation between quantities of this kind; and vice verſâ, any equations expreſſing

the relations between quantities of this kind may in like manner be tranſlated into algebraical, &c. equations.

9. Hence any algebraical equations containing (l) unknown quantities, may be tranſlated into equations expreſſing the relation between $(l-r)$ unknown quantities and (r) different independent chances; and theſe laſt mentioned equations may be tranſlated into the former.

10. Any given equations expreſſing the relations between algebraical, &c. quantities may, by the preceding methods, be tranſlated into innumerable different equations; for every unknown quantity, its powers, &c. contained in the given equations, may be ſubſtituted the chances or probabilities of an event's ſucceeding for certain number of times, or its not ſucceeding; or a compound function of both; and the reſolutions, reductions, &c. of the reſulting equations can eaſily be deduced from the reſolutions, reductions, &c. of the given ones.

11. e. g. Let the chance of an event's ſucceeding, contained in the given equations, be $\frac{a}{N}$, then if all the terms of the given equations for a be ſubſtituted $N-Na'$, there will reſult equations, in which a is exterminated, and a' the chance of its not ſucceeding is ſubſtituted in its place: but if in ſome of the terms of the equation be ſubſtituted $N-N\times a'$ for a, and in

others not, there will reſult different equations, in which are contained both a and a', &c.

12. If the ſame algebraical quantities, &c. are expreſt in different algebraical, &c. formulæ, then by tranſlating them into probable relations, as is before taught, the equality will be tranſlated into a propoſition concerning chances.

Ex. 1. $\frac{a}{N} \times \frac{N-a}{N} = \frac{a}{N} - \frac{a^2}{N^2}$; let $\frac{a}{N}$ be the chance of an event's ſucceeding in one trial, then will $\frac{N-a}{N}$ be the chance of its failing, and $\frac{a^2}{N^2}$ the chance of its happening twice together; and therefore the chance of its happening in the firſt trial and failing in the ſecond will be equal to the difference between the chance of the event's ſucceeding in the firſt trial and its ſucceeding twice together: but if the order of its happening and failing is not given, then the one half of the chance of its happening once and failing once in two trials will be equal to the abovementioned difference.

Ex. 2. $\frac{a+b}{N} \times \frac{a-b}{N} = \frac{a^2-b^2}{N^2}$, and conſequently the chance of an event's happening, of which the probability is $\frac{a+b}{N}$, and then an event happening, of which the probability is $\frac{a-b}{N}$, is equal to the difference between the chances of two different events reſpectively happening twice in two trials, of which their reſpective chances for happening in one trial are $\frac{a}{N}$ and $\frac{b}{N}$.

Ex. 3. $$\frac{\overline{a+b}^{n}}{N^{n}} = \frac{a^{n} + na^{n-1}b + n\,.\,\frac{n-1}{2}\,a^{n-2}b^{2} \ldots\ldots + n\,.\,\frac{n-1}{2}\,a^{2}b^{n-2} + nab^{n-1} + b^{n}}{N^{n}}$$, and conſequently the chance of an event's happening n times in n trials, of which the chance of its happening in one trial is $\frac{a+b}{N}$, will be equal to the ſum of the chances of two events A and B happening n times in n trials, and of the chances of the one happening $n-1$ times and the other one time in the abovementioned trials, and of the chances of the one happening $n-2$ times and the other two times in the ſame trials, &c. when the order of the event's happening is not given; the chances of the events A and B happening in one trial being reſpectively $\frac{a}{N}$ and $\frac{b}{N}$: but if the order of the abovementioned events be given, then will the abovementioned chance of an event's (of which the chances of happening in one trial is $\frac{a+b}{N}$) happening n times in n trials

n trials will be equal to the ſum of the chances A and B happening in n trials $+ n \times$ ſum of the chances of the one happening $n-1$ times and the other one time $+ n \times \frac{n-1}{2} \times$ ſum of the one happening $n-2$ and the other two times, &c. in the ſame number (n) of trials.

Ex. 4. $\frac{N}{N-a} = 1 + \frac{a}{N} + \frac{a^2}{N^2} + \frac{a^3}{N^3} +$ &c. in infinitum; but $\frac{N-a}{N}$ is the probability of an event's failing, of which the probability of its happening is $\frac{a}{N}$; therefore the reciprocal of the probability of an event's failing, of which the probability of its happening is $\frac{a}{N}$, will exceed 1 by the ſum of the probabilities of the abovementioned events happening once in one trial, twice in two trials, three times in three trials, four times in four, &c. in infinitum.

Ex. $\frac{e}{a+b} = \frac{e}{a} - \frac{e}{a}\left(\frac{b}{a} - \frac{b^2}{a^2} + \frac{b^3}{a^3} - \frac{b^4}{a^4} + \right.$ &c.); ſuppoſe three events A, B and C, of which the reſpective chances for happening in each trial are $\frac{e}{a+b}$, $\frac{e}{a}$ and $\frac{b}{a}$; then the chance of A's happening will be leſs than the chance of B's happening by the chance of B's happening once into the difference between the chances of C's happening 1, 3, 5, 7, &c. times and its happening 2, 4, 6, &c. times in infinitum.

Quantities of this kind may be interpolated by interpolating the quantities contained in them, their indexes, &c.; they may be multiplied and divided by each other, proper attention being had to the mode of tranſlation before given.

If two or more different chances may juſtly be tranſlated into the ſame algebraical quantity, they may be concluded equal.

A chance may conſiſt of the ſum or difference of ſeveral chances, which may depend on the happening of ſeveral events, and be tranſlated accordingly.

It may not be unworthy of obſervation, that the tranſlation of quantities into their different equalities or meanings, and decompoſing them by different methods into the different parts of which they conſiſt, conſtitute a conſiderable ſhare of many ſciences. In arithmetic the bare expanſion of terms often develops many propoſitions before concealed: e. g. Let the chances of two events (A and B) happening be to each other :: $a : b$; and the ſum of all the chances poſſible, i. e. certainty N; then will the chance of the events A and B happening n times in n trials be $\frac{\overline{a+b}^{\,n}}{N^n}$

$$= \frac{a^n + n a^{n-1} b + n . \frac{n-1}{2} a^{n-2} b^2 + \ldots + L a^{n-l} b^l +}{N^n}$$

$$\underline{\therefore+Ma^{n-m}b^{m}+\ldots+n.\frac{n-1}{2}\times a^{2}b^{n-2}+nab^{n-1}+b^{n}},$$

of which the chance of happening in one trial is $\frac{a+b}{N}$.

2. The chance of the events (A)'s happening $(n-l)$ times and (B)'s happening (l) times will be the term $L\times\frac{a^{n-l}\times b^{l}}{N^{n}}$; and the chance of (A)'s happening $(n-l)$ times or more in (n) trials will be the ſum of all the terms to $L\times a^{n-l}\times b^{l}$ included, divided as before by N^{n}; and the chance of (A)'s happening fewer times than $(n-l)$ and B's happening more times than (l) will = the remaining terms divided by the ſame diviſor.

3. The chance of the event (A) happening an even number to its happening an odd number of times, will be as the ſum of all the alternate terms to each other; in a ſimilar manner may be deduced the chance of its happening every hth time to the chance of its happening kth time.

4. The chance of (A)'s gaining $(n-l)$ games to B's gaining $(l-1)$ games will be as $a^{n}+na^{n-1}b+n.\frac{n-1}{2}a^{n-2}b^{2}+\ldots+La^{n-l}b^{l}$ to the ſum of all the remaining terms; and from the ſame principles may be deduced the chance of the events (A)'s happening between h and k times to A's or B's happening between l and m times.

5. The chance of *A*'s happening once, twice, thrice, or $\frac{h}{k}$ times for *B*'s happening once in n trials will $= L \times \frac{a^{n-l} \times b^{l}}{N^{n}}$, where $n-l = l$, $2l$, $3l$, or $\frac{h}{k} \times l$; and the chance of its happening the abovementioned number of times or more to its not happening $:: a^{n} + n a^{n-1} b + \ldots L a^{n-l} b^{l}$ to the ſum of all the remaining terms.

6. The chance of *A*'s winning r games more than B to the chance of his not doing it $:: a^{n} + n a^{n-1} b + \ldots + L a^{n-l} b^{l}$: ſum of the remaining terms, where $l = \frac{n-r}{2}$.

Innumerable other like propoſitions may eaſily be deduced.

If any of the preceding quantities are required to be equal, or in a given ratio to each other; ſuppoſe them equal, &c. and from the reſulting equation or equations, by the principles of algebra, approximations, &c. inveſtigate the unknown quantities.

The number of unknown quantities, as is well known, muſt not exceed the number of equations.

In theſe problems, the unknown quantities mentioned are a, b, n and N, and if three of them are given, the fourth can be found; if n be the quantity required, the equation will become an exponential one; if the proportion of $a:b$, and not the quantities themſelves, be required, for a ſubſtitute 1.

2. Let more quantities a, b, c, d, &c. be the reſpective chances of any different events A, B, C, D, &c. to certainty N; and $\frac{a+b+c+d+\&c.}{N}$ the chance of an event's happening; then will the chance of its happening n times in n trials be $\left(\frac{a+b+c+d+\&c.}{N}\right)^n$

$$= \frac{a^n + n a^{n-1}(b+c+d+\&c.) + n.\frac{n-1}{2} \times a^{n-2}(b^2 + c^2 + d^2 + \&c.) + n.\overline{n-1} \times a^{n-2}(bc + bd + cd + \&c.) + \ldots + L a^l b^m c^p d^q \&c. + \&c.}{N^n}$$

; and the chance of A's happening l times, B's happening m times, C's happening p times, D's happening q times, &c. will then be

$$n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-l+1}{l} \times \frac{n-l}{1} \times \frac{n-l-1}{2} . \frac{n-l-2}{3} \ldots \frac{n-l-m+1}{m} \times \frac{n-l-m}{1} \times \frac{n-l-m-1}{2} \ldots \times \frac{n-l-m-p+1}{p} \times \frac{n-l-m-p}{1} \times \frac{n-l-m-p-1}{2} \ldots \frac{n-l-m-p-q+1}{q} \times \&c.\ (L) \times a^l b^m c^p d^q \&c.;$$

where $n = l+m+p+q+\&c.$ divided by N^n.

Every corollary contained in the preceding caſe may be applied to this, and in the terms which include a, b, c, d, &c. equally, as a and b.

Cor. Let $\left(\frac{a+b+c+d+\&c.}{N}\right)^n = \left(\frac{a^1+a^2+a^3 \ldots + a^m}{N}\right)^n = \frac{a^n + n\,a^{n+1} + \ldots + L \times a^l \ldots + a^{mn}}{N^n}$, and conſequently if there are (m) events, of which the probabilities of ſucceeding are equal and $\frac{p}{N}$, marked 1, 2, 3, 4, . . m points, then will the chance of the happening of l points in n trials be $\frac{Lp^n}{N^n}$.

3. If ſome terms favour two or more parties, decompoſe them into all their ſimple parts, and add the parts favouring any one to his expectation. e. g. ſuppoſe the term $a^m b^{l-1}$; and let A (whoſe chance for ſucceeding at every one trial is a) gain, if he ſucceeds r times before B (whoſe chance for winning at every trial is b) ſucceeds s times, where r and s are equal or leſs than m and $l-1$ reſpectively; and C ($a^h b^k$) denote the number of all the different combinations, into which the quantity $a^h b^k$ can be decompoſed, viz. $a^h b^k$, $a^{h-1} b^k a$, $a^{h-2} b^{k-1} a\,b\,a$, $a^{h-2} b^k a^2$, &c. of which the number will be $\overline{h+k} \times \frac{h+k-1}{2} \times \frac{h+k-2}{3} \ldots \frac{h+1}{k}$ or $\frac{k+1}{h}$; then will the number of terms,

in which *A* gains be $C(a^m b^{s-1}) \times b^{l-s} + C(a^{m-1} b^{s-1}) \times C(a b^{l-s}) + C(a^{m-2} b^{s-1}) \times C(a^2 b^{l-s}) + C(a^{m-3} b^{s-1}) \times C(a^3 l^{l-s}) \ldots\ldots \times C(a^r b^{s-1}) \times C(a^{m-r} b^{l-s})$; ſince in each of theſe combinations *A* gains (r) times before *B* gains (s) times; but the number of theſe combinations are

$$\begin{gathered}\overline{m+s-1} \times \frac{m+s-2}{2} \times \frac{m+s-3}{3} \cdots \frac{s}{m} \left(\text{or } \frac{m+1}{s-1}\right) \times 1 + \overline{m+s-2} \times \\ \frac{m+s-3}{2} \cdot \frac{m+s-4}{3} \cdots \frac{s}{m-1} \left(\text{or } \frac{m}{s-1}\right) \times \\ \overline{l-s+1} + \overline{m+s-3} \cdot \frac{m+s-4}{2} \cdot \frac{m+s-5}{3} \cdots \\ \frac{s}{m-2} \left(\text{or } \frac{m-1}{s-1}\right) \times \overline{l-s+2} \cdot \frac{l-s+1}{2} + \overline{m+s-4} \\ \cdot \frac{m+s-5}{2} \cdot \frac{m+s-6}{3} \cdots \frac{s}{m-3} \left(\text{or } \frac{m-2}{s-1}\right) \times \\ \overline{l-s+3} \times \frac{l-s+2}{2} \times \frac{l-s+1}{3} + \cdots + \overline{r+s-1} \\ \times \frac{r+s-2}{2} \cdot \frac{r+s-3}{3} \cdots \frac{s}{r} \left(\text{or } \frac{r+1}{s-1}\right) \times \overline{l-s+1} \\ \frac{l-s+2}{2} \times \frac{l-s+3}{3} \cdots \frac{l-s+1}{m-r} \left(\text{or } \frac{m-r+1}{l-s}\right).\end{gathered}$$

If in the above quantity for m and r be ſubſtituted $l-1$ and s; and vice verſâ for $l-1$ and s be ſubſtituted m and r, the quantity reſulting will expreſs the number of terms which favour *B*.

1. If

4. 1. If the chance depends on the expectations contained in a number of ſucceſſive trials: Find the chance of the expectation from the firſt trials; then find the chance of its failing in the firſt and ſucceeding in the ſecond, and ſo on in infinitum; or the chance of its ſucceeding in the ſecond to be added to the firſt, and ſo on; the ſum of the reſulting chances will be the chance required.

2. If the ſeveral expectations depend on the chances of ſome events happening and others failing or happening in ſome given relation to them: Find the chances of each of the events happening and the other failing or happening in the given relation; and ſo of each of the expectations; and thence the chance required.

Ex. 1. Let the chances of the events A and B happening be reſpectively $\frac{a}{a+b}$ and $\frac{b}{a+b}$; then the chance of the event A happening r times more than b in r trials will be $\frac{a^r}{(a+b)^r}$; in $r+2$ trials will be $\frac{a^r}{(a+b)^r}\left(1+r\times\frac{ab}{(a+b)^2}\right)$; in $r+4$ trials will be $\frac{a^r}{(a+b)^r}\left(1+r\times\frac{ab}{(a+b)^2}+r.\frac{r+3}{2}\times\frac{a^2b^2}{(a+b)^4}\right)$, and in general it will be $\frac{a^r}{(a+b)^r}\times$

$$(1+r\times\frac{ab}{(a+b)^2}+r\,.\,\frac{r+3}{2}\times\frac{a^2b^2}{(a+b)^4}+r\,.\,\frac{r+4}{2}\,.\,\frac{r+5}{3}\times\frac{a^3b^3}{(a+b)^6}+r\,.\,\frac{r+5}{2}\,.\,\frac{r+6}{3}\,.\,\frac{r+7}{4}\times\frac{a^4b^4}{(a+b)^8}+\ldots+r\,.\,\frac{r+l+1}{2}\,.\,\frac{r+l+2}{3}\,.\,\frac{r+l+3}{4}\,.\,.\,.\,.\,.\frac{r+2l-1}{l}\times\frac{a^l b^l}{(a+b)^{2l}}+\ldots+\text{\&c. in infinitum})\text{.}$$

This may be deduced from the ſubſequent arithmetical theorem. viz.

$$2m\times\frac{2m-1}{2}\,.\,\frac{2m-2}{3}\,.\,.\,.\,\frac{2m-s}{s+1}+r\times(2m-2)\,.\,\frac{2m-3}{2}\,.\,\frac{2m-4}{3}\,.\,.\,.\,\frac{2m-s-1}{s}+r\,.\,\frac{r+3}{2}\times(2m-4)\,.\,\frac{2m-5}{2}\,.\,.\,.\,\frac{2m-s-2}{s-1}+r\,.\,\frac{r+4}{2}\,.\,\frac{r+5}{3}\times(2m-6)\,.\,\frac{2m-7}{2}\,.\,\frac{2m-8}{3}\,.\,.\,\frac{2m-s-3}{s-2}+\ldots+r\,.\,\frac{r+s+2}{2}\,.\,\frac{r+s+3}{2}\,.\,\frac{r+s+4}{3}\,.\,.\,.\,\frac{r+2s+1}{s+1}=(r+2m)\,.\,\frac{r+2m-1}{2}\,.\,\frac{r+2m-2}{3}\,.\,\frac{r+2m-3}{4}\,.\,.\,\frac{r+2m-s}{s+1}\,.$$

Ex. 2. If it be required to find the chance of *A*'s ſucceeding *n* times as oft as *B*'s preciſely: in n+1 trials it will be found $(n+1)\,\frac{a^n b}{(a+b)^{n+1}}=P$;

$= P$; in $2n+2$ trials it will be found $P + n \cdot (n+1) \times \frac{a^{2n}b^{2}}{(a+b)^{2n+2}} = \mathcal{Q}$; in $3n+3$ it will be $\mathcal{Q} + n \cdot \frac{n+1}{2} \cdot (3n+1) \times \frac{a^{3n}b^{3}}{(a+b)^{3n+3}} \times$ &c. The ſame principles may be applied to innumerable examples, and ſimilar propoſitions may be collected (mutatis mutandis) when applied to the caſe contained in the following theor.

5.1. Let a, b, c, d, &c. be the reſpective chances of the happening of $\alpha, \beta, \gamma, \delta$, &c.: in one trial (A), and $(ax^{\alpha} + bx^{\beta} + cx^{\gamma} + dx^{\delta} +$ &c.$)^{n} = a^{n}x^{n \times \alpha} + \ldots\ldots + Nx^{\pi} +$ &c.; then will N be the chance of the happening of π in n trials.

2. Let r, s, t, &c. be the reſpective chances of the happening of ρ, σ, τ, &c. in one trial (B), and $(ax^{\alpha} + bx^{\beta} + cx^{\gamma} +$ &c.$)^{n} \times (rx^{\rho} + sx^{\sigma} + tx^{\tau} +$ &c.$)^{m} = a^{n}r^{m}x^{n\alpha + m\rho} + \ldots N'x^{\pi'} +$ &c.; then will N' be the chance of π' happening in n trials of A, and m trials of B.

The ſame principles may be applied to more quantities, &c.

From ſome data in the Meditat. Analyt. and Philoſ. Tranſ. may be deduced other ſimilar propoſitions.

THEOR.

1. $\overline{a+b}\,.\,\overline{a+b-1}\,.\,\overline{a+b-2}\,.\,\overline{a+b-3}\ldots$
$\overline{a+b-n+1} = a\,.\,\overline{a-1}\,.\,\overline{a-2}\ldots\overline{a-n+1}+$
$n\times a\,.\,\overline{a-1}\,.\,\overline{a-2}\ldots\overline{a-n+2}\times b+n\,.\,\frac{n-1}{2}$
$.\,a\,.\,\overline{a-1}\,.\,\overline{a-2}\ldots\overline{a-n+3}\times b\,.\,\overline{b-1}+n\,.$
$\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,a\,.\,\overline{a-1}\,.\,\overline{a-2}\,..\,\overline{a-n+4}\,.\,b\,.\,\overline{b-1}$
$.\,\overline{b-2}+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\frac{n-3}{4}\,.\,a\,.\,\overline{a-1}\,.\,\overline{a-2}$
$\ldots\overline{a-n+5}\times b\,.\,\overline{b-1}\,.\,\overline{b-2}\,.\,\overline{b-3}+\ldots$
$+L\times a\,.\,\overline{a-1}\,.\,\overline{a-2}\,..\,\overline{a-n+l}\times b\times\overline{b-1}\,.$
$\overline{b-2}\,.\,\overline{b-3}\ldots\overline{b-l+2}\ldots+M\times a\,.\,\overline{a-1}\ldots$
$\overline{a-n+m}\times b\,.\,\overline{b-1}\,.\,\overline{b-2}\,..\,\overline{b-m+2}+\ldots+$
$n\,.\,\frac{n-1}{2}\,.\,a\,.\,\overline{a-1}\,.\,b\,.\,\overline{b-1}\,.\,\overline{b-2}\,.\,\overline{b-3}\ldots$
$\overline{b-n+3}+n\,.\,a\times b\,.\,\overline{b-1}\,.\,\overline{b-2}\ldots\overline{b-n+2}$
$+b\,.\,\overline{b-1}\,.\,\overline{b-2}\,.\,\overline{b-3}\ldots\overline{b-n+1}.$

2. If for the terms $a+b-1$, $a+b-2$, $a+b-3$, &c.; $a-1$, $a-2$, $a-3$, &c.; $b-1$, $b-2$, $b-3$, &c. be ſubſtituted $a+b-x$, $a+b-2x$, &c.; $a-x$, $a-2x$, &c.; $b-x$, $b-2x$, &c.; the equation reſulting will be equally juſt: when $x=0$, it becomes the binomial theorem.

3. Let a and b be the chances of the two

events, of which the ſum of all the events poſſible or certainty is N, and after the firſt trial let the ſum of events or certainty be diminiſhed by 1, i. e. let it be $N-1$, and alſo the numbers of events to happen (a and b) or to be drawn, be diminiſhed by 1; and in the ſame manner, after the ſecond trial, let N be diminiſhed by 2, and the number of events to happen (a and b) be ſtill diminiſhed by 1; that is, let it be $a-2$, if a happened the firſt and ſecond time; or $a-1$ and $b-1$, if a and b happened each once only; and ſo on: then will the chance of the event (of which the chance of happening the firſt time is $\frac{a+b}{N}$) happening n times together be $\frac{\overline{a+b}\times\overline{a+b-1}\,.\,\overline{a+b-2}}{N\,.\,\overline{N-1}\,.\,\overline{N-2}}$

$$\frac{\ldots,\overline{a+b-n+1}}{\ldots\overline{N-n+1}} = \underline{a\,.\,\overline{a-1}\,.\,\overline{a-2}\,..\,\overline{a-n+1}}$$

$$\underline{+\,n\times a\,.\,\overline{a-1}\,.\,\overline{a-2}\,..\,\overline{a-n+2}\times b+n\,.\,\frac{n-1}{2}}$$

$$N\,.\,\overline{N-1}\,.\,\overline{N-2}\,.\,\overline{N-3}\,.$$

$$\underline{\times\,a\,.\,\overline{a-1}\,.\,\overline{a-2}\,...\,\overline{a-n+3}\times b\,.\,\overline{b-1}+n\,.\,\frac{n-1}{2}\,.}$$

$$\overline{N-4}\times\ldots\overline{N-n+1}$$

$$\underline{\frac{n-2}{3}\,a\,.\,\overline{a-1}\,.\,\overline{a-2}\,...\,\overline{a-n+4}\times b\,.\,\overline{b-1}\,.\,\overline{b-2}}$$

$+\ldots$

$$\underline{+\ldots L\times a\,.\,\overline{a-1}\,.\,\overline{a-2}\,.\,.\,\overline{a-n+l}\times b\,.\,\overline{b-1}\,.}$$

$$\underline{\overline{b-2}\ldots\overline{b-l+2}\,\ldots\, na\,.\,b\,.\,\overline{b-1}\,.\,\overline{b-2}\,.}$$

$$\underline{\overline{b-n+2}+b\,.\,\overline{b-1}\,.\,\overline{b-2}\,.\,.\,\overline{b-n+1}}$$

: it appears that the chance of the event's, of which the probability is $\frac{b}{a+b}$, happening $l-1$ times in (n) trials, and no more, will be $\frac{L\times a\,.\,\overline{a-1}\,.\,\overline{a-2}}{\overline{a+b}}$.

$$\frac{\ldots\overline{a-n+l}\times b\,.\,\overline{b-1}\,.\,\overline{b-2}\ldots\overline{b-l+2}}{\overline{a+b-1}\,.\,:\,.\,\overline{a+b-n+1}}.$$

From hence it follows, that every thing aſſerted of the former caſe may equally be applied to this, by only ſubſtituting for $a^l\times b^k$ in the former caſe the content $a\,.\,\overline{a-1}\,.\,\overline{a-2}\ldots\overline{a-l+1}\times b\times\overline{b-1}\times\overline{b-2}\ldots\overline{b-k+1}$, and for N^n the content $N\,.\,\overline{N-1}\,.\,\overline{N-2}\,.\,.\,\overline{N-n+1}$.

3. When more quantities a, b, c, d, &c. are contained in trials of this kind, then every thing that is aſſerted in the former caſe may alſo equally be applied to this, for the former will be tranſformed into this by writing in it for a^l, the content $a\times\overline{a-1}\times\overline{a-2}\,.\,.\,\overline{a-l+1}$, and ſimilarly for b^k, c^p, d^q, &c. reſpectively, the contents b .

$b-1$

$\overline{b-1}\,.\,\overline{b-2}\,..\,\overline{b-k+1}$, $c\times\overline{c-1}\,.\,\overline{c-2}\,..\,\overline{c-p+1}$, $d\,.\,\overline{d-1}\,.\,\overline{d-2}\,..\,\overline{d-q+1}$: the coefficients of the terms in both caſes are the ſame, i. e. the coefficient of the term $a\,.\,\overline{a-1}\,.\,\overline{a-2}\,\ldots\,\overline{a-l+1}\times b\,.\,\overline{b-1}\,.\,\overline{b-2}\,..\,\overline{b-k+1}\times c\,.\,\overline{c-1}\,.\,\overline{c-2}\ldots\overline{c-p+1}\times d\times\overline{d-1}\,.\,\overline{d-2}\ldots\overline{d-q+1}\times$ &c. will be $n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\ldots\frac{n-l+1}{l}\times\overline{n-l}\,.\,\frac{n-l-1}{2}\,.\,\frac{n-l-2}{3}\,..\,\frac{n-l-k+1}{k}\times\overline{n-l-k}\,.\,\frac{n-l-k-1}{2}\times\frac{n-l-k-2}{3}\ldots\frac{n-l-k-p+1}{p}\times\overline{n-l-k-p}\times\frac{n-l-k-p-1}{2}\times\frac{n-l-k-p-2}{3}\,..\,\frac{n-l-k-p-q+1}{q}\times$ &c. in the ſeries for $(a+b+c+d+\&c.)\times(a+b+c+d+\&c.-1)\times(a+b+c+d+\&c-2)\times(a+b+c+d+\&c.-3)\,..\,a+b+c+d+\&c.-n+1)=a\times(a-1)\,.\,(a-2)\,.\,(a--n+1)+na\,.\,(a-1)\,..\,(a-n+2)\times b+n\,.\,\frac{n-1}{2}\,a\,.\,(a-1)\,..\,(a-n+3)\times b\,.\,(b-1)+$&c. which is alſo the coefficient of the term $a^l\,b^k\,c^p\,d^q$ &c. in the power n of the multinomial $a+b+c+d+$&c. expanded, i. e. $(a+b+c+d+\&c.)^n=a^n+na^{n-1}\,b+n\,.\,\frac{n-1}{2}\,a^{n-2}\,b^2+$&c.

Cor. The ſum of the contents of every $n-s$ of the $(n-1)$ numbers $1, 2, 3, 4, \ldots n-1 \times s \cdot \frac{s-1}{2} \cdot \frac{s-2}{3} \ldots \frac{s-r+1}{r} = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \ldots \frac{n-r+1}{r} (H)$ × ſum of the contents of every $(n-s)$ of the numbers $1, 2, 3, 4, \ldots n-r-1 + H \times \frac{n-r}{r+1} (K)$ × ſum of the contents of every $(n-s-1)$ of the numbers $1, 2, 3, \ldots n-r-2$ × ſum of the numbers $1, 2, 3, \ldots r + K \times \frac{n-r-1}{r+2} (L)$ × ſum of the contents of every $(n-s-2)$ of the numbers $1, 2, 3, \ldots n-r-3$ × ſum of the products of every two of the numbers $1, 2, 3, \ldots r, r+1, + L \times \frac{n-r-2}{r+3} (M)$ × ſum of the contents of every $(n-s-3)$ of the numbers $1, 2, 3, \ldots n-r-4$ × ſum of the contents of every three of the numbers $1, 2, 3, \ldots r, r+1, r+2$ + &c. s being leſs than n.

ON ANNUITIES.

Let n, n', n'', n''', &c. be the numbers of perſons living according to any bills of mortality of the ages of N, $N+1$, $N+2$, $N+3$, &c. years, and m, m', m'', m''', &c. the number of perſons living of the ages of M, $M+1$, $M+2$, $M+3$, &c. years, and p, p', p'', p''', &c. the number of perſons living at the reſpective ages of P, $P+1$, $P+2$. $P+3$, &c. years; a ſimilar notation may be uſed for the reſpective ages of $\mathcal{Q}$, $\mathcal{Q}+1$, $\mathcal{Q}+2$, &c.; R, $R+1$, &c.; S, $S+1$, &c.; $\mathcal{T}$, $\mathcal{T}+1$, &c. years.

1. The chance of a perſon of N years of age living one year according to theſe tables will be $\frac{n'}{n}$ (or number of perſons living at the end of the year divided by the number of perſons living at the preſent time); of his living two, three, ... v years will be reſpectively $\frac{n''}{n}$, $\frac{n'''}{n}$, ..., $\frac{n^v}{n}$.

2. If $\frac{x}{n}$ be the chance of a perſon's living for any time; then $\frac{x}{n}$ ſubtracted from certainty (1) will be the chance of his dying in the ſame time, viz. $\frac{n-x}{n}$.

3. Let $r-1$ be the intereſt of one pound for one year, and the preſent value of any ſum of money L to be paid at the end of v years on the happening or failing of certain conditions in that time, of which the chance of their happening or failing is W, will be $L\times\frac{W}{r^v}$; e. g. the chance of a perſon of N years of age living one year is $\frac{n'}{n}$, and the preſent value of his expectation of L pounds at the end of the year is $L\times\frac{n'}{n\times r}$; at the end of two years $L\times\frac{n''}{nr^2}$, &c.

4. Let the preſent value conſiſt of ſeveral expectations; from the preceding principles find the value of each of the expectations and add them together: e. g. the preſent value of an annuity L for the life of a perſon of N years of age at $r-1$ rate of intereſt from theſe principles will be found $L\times\left(\frac{n'}{nr}+\frac{n''}{nr^2}+\frac{n'''}{nr^3}+\right.$ &c. to the extremity of life).

Ex. 2. The preſent value of the perpetuity of L pounds annually, to commence at the beginning of the year immediately after the death of a perſon of the age of N will be $L\times\left(\frac{n-n'}{n.r}+\frac{n-n''}{nr^2}+\frac{n-n'''}{nr^3}+\text{\&c.}\right)=L\left(\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\right.$

$\frac{1}{r^4}$ + &c.) preſent value of the perpetuity — $L \times \left(\frac{n'}{nr} + \frac{n''}{nr^2} + \frac{n'''}{nr^3}\right.$ + &c.) preſent value of the life annuity.

Ex. 3. To find the preſent value of an annuity (L) for the joint lives of perſons of the ages N, M, P, Q, &c.

The chance of a perſon living to the end of any year will be from the tables = the number of perſons living to the end of that year divided by the number of perſons living at the preſent time, and conſequently the chances of perſons of the ages N, M, P, Q, &c. living to the end of the firſt year will be reſpectively $\frac{n'}{n}$, $\frac{m'}{m}$, $\frac{p'}{p}$, $\frac{q'}{q}$, &c.; and their chances of living to the end of the ſecond year will be $\frac{n''}{n}$, $\frac{m''}{m}$, $\frac{p''}{p}$, $\frac{q''}{q}$, &c. reſpectively, &c.; and conſequently the chance of their joint lives continuing the firſt year will be $\frac{n'}{n} \times \frac{m'}{m} \times \frac{p'}{p} \times \frac{q'}{q} \times$ &c.; and the chance of their joint lives continuing the ſecond year will be $\frac{n''}{n} \times \frac{m''}{m} \times \frac{p''}{p} \times \frac{q''}{q} \times$ &c.; and ſo on; whence the preſent value of the annuity for their joint lives at $r-1$

rate of intereſt will be $\frac{L}{r}\left(\frac{n'}{n}\times\frac{m'}{m}\times\frac{p'}{p}\times\frac{q'}{q}\times\text{\&c.}\right)$ $+\frac{L}{r^2}\left(\frac{n''}{n}\times\frac{m''}{m}\times\frac{p''}{p}\times\frac{q''}{q}\times\text{\&c.}\right)+\frac{L}{r^3}\left(\frac{n'''}{n}\times\frac{m'''}{m}\right.$ $\left.\times\frac{p'''}{p}\times\frac{q'''}{q}\times\text{\&c.}\right)+\text{\&c.}$ to the complement of the oldeſt life.

Ex. 4. To find the preſent value of an annuity (L) to be paid on condition, that a number of perſons N, M, &c. be living, and ſome other perſons P, Q, &c. be dead.

The preſent value of the annuity will be

$$\frac{L}{n\,.\,m\,.\,p\,.\,q\,.\,\text{\&c.}}\left(\frac{n'.\,m'.\,\text{\&c.}\times(p-p')\,.\,(q-q')\,.\,\text{\&c.}}{r}\right.$$

$$+\frac{n''\,.\,m''\,.\,\text{\&c.}\times(p-p'')\,.\,(q-q'')\,.\,\text{\&c.}}{r^2}+$$

$$\frac{n'''\,.\,m'''\,.\,\text{\&c.}\times(p-p''')\,.\,(q-q''')\,.\,\text{\&c.}}{r^3}+\text{\&c.}$$

This ſeries muſt terminate at the complement of the oldeſt life of the perſons N, M, &c., which ſuppoſe N, for the condition of the perſons N, M, &c. living neceſſarily ends at that time; it muſt alſo change at the complement of life of P, if P be the oldeſt life, and older than N for then P muſt neceſſarily die, and the condition becomes that of N, M, &c. living and Q, &c. dying; and there muſt occur a change in the ſeries at the complement of every life older than N; proper diſcount being allowed in the new ſerieſes from the preſent time to the complements of life.

Ex. 5. To find the preſent value of an annuity for the longeſt liver of perſons, of which the ages are reſpectively *N*, *M*, *P*, *Q*, &c. years at $r-1$ rate of intereſt.

The chances of the perſons of the ages *N*, *M*, *P*, *Q*, &c. dying in the firſt year are reſpectively $1-\frac{n'}{n}$, $1-\frac{m'}{m}$, $1-\frac{p'}{p}$, $1-\frac{q'}{q}$, &c.; and conſequently the chance of all of them dying in the firſt year will be $\left(1-\frac{n'}{n}\right)\times\left(1-\frac{m'}{m}\right)\times\left(1-\frac{p'}{p}\right)$ $\times$ &c. $=1-\left(\frac{n'}{n}+\frac{m'}{m}+\frac{p'}{p}+\text{\&c.}\right)+\left(\frac{n'm'}{nm}+\frac{n'p'}{np}+\frac{m'p'}{mp}+\text{\&c.}\right)-\left(\frac{n'm'p'}{nmp}+\text{\&c.}\right)+$ &c.; which ſubtracted from certainty (1) will leave the chance of ſome one or more living during the firſt year $=\left(\frac{n'}{n}+\frac{m'}{m}+\frac{p'}{p}+\text{\&c.}\right)-\left(\frac{n'm'}{nm}+\frac{n'p'}{np}+\frac{m'p'}{mp}+\text{\&c.}\right)+\left(\frac{n'm'p'}{nmp}+\text{\&c.}\right)-$ &c. and divided by *r* gives the preſent value of the chance of the annuity payable the firſt year; and in a ſimilar manner it may be deduced for every ſubſequent year; whence the preſent value of the annuity will be found = ſum of the values of annuities for all the ſeparate lives of perſons of the ages of N, M, P, Q, &c. — ſum of all the values of annuities for every two joint lives of the abovementioned perſons + ſum of the values of annuities for every three joint lives — ſum of all the values of annuities for every four joint lives of the above mentioned perſons + &c.

Ex. 6. To find the preſent value of an annuity, on the ſuppoſition that S pounds is to be paid at the end of the year, if no perſon is dead, S' if one perſon only is dead, S'' if two perſons, S''' if three, &c., and the number of perſons to be (l), of which the ages are reſpectively N, M, P, Q, &c.

Let $A =$ value of the annuity of 1£. for the joint lives of all the abovementioned perſons, of which the ages are N, M, P, Q, &c.; A' the aggregate of the values of the joint lives of each (l—1) perſons abovementioned; A'', A''', A'''', &c. the reſpective aggregates of the values of the joint lives of each l—2, l—3, l—4, &c. of the ſame perſons; then will the annuity required be.

$$A \times S + (A' - l \times A) \times S' + (A'' - (l-1) \times A' + l \,.\, \frac{l-1}{2} A) \times S'' + (A''' - (l-2) \times A'' + (l-1) \,.\, \frac{l-2}{2} \times A' - l \,.\, \frac{l-1}{2} \,.\, \frac{l-2}{3} \times A) \times S''' + (A''' -$$

$$\left(A'''' - (l-3) \times A''' + (l-2) \times \frac{l-3}{2} A'' - (l-1) . \frac{l-2}{2} . \frac{l-3}{3} \times A' + l . \frac{l-1}{2} . \frac{l-2}{3} . \frac{l-3}{4} \times A\right) \times S'''' + \&c.$$

This will appear from multiplying the preſent chance of each $(l-b)$ perſons N, M, &c. living for each year, viz. $\frac{n'm'}{nm}$ &c., $\frac{n''m''}{nm}$ &c. &c. into the chance of the (b) remaining perſons dying in the ſame year.

Cor. Let A denote the chance of the (l) perſons living for any time, A' the aggregate of the chances of each $(l-1)$ of the abovementioned perſons living the ſame time; A'' the aggregate of the chances of each $(l-2)$ abovementioned perſons living the ſame time; and in the ſame manner let A''', A'''', &c. denote the reſpective aggregates of the chances of each $l-3$, $l-4$, &c. living in the ſame time; then will the chance of all the (l) perſons living during that time $= A$ as before, the chance of all but any one living during that time $= A' - l \times A$, the chance of all but any two living as beforementioned $= A'' - (l-1)\, A' + l . \frac{l-1}{2} \times A$, the chance of all but any three living to that time $= A''' - (l-2) \times A'' + (l-1) . \frac{l-2}{2} A' - l . \frac{l-1}{2} . \frac{l-2}{3} \times A$, the chance of all but any four living as before $= A'''' - (l-3) \times A''' + (l-2) \cdot \frac{l-3}{2} \times A'' - (l-1) \cdot \frac{l-2}{2} \cdot \frac{l-3}{3} \times A' + l \cdot \frac{l-1}{2} \cdot \frac{l-2}{3} \cdot \frac{l-3}{4} \times A$; &c.

Cor. Let B' be the aggregate of the chances of each perſon N, M, P, Q, &c. living a given time; B'' the aggregate of the chances of every two abovementioned joint lives for that time; B''' the aggregate of the chances of every three joint lives of the perſons N, M, P, Q, &c. abovementioned for the ſame time; B'''' the aggregate of the chances of every four joint lives of them for the ſame time, &c. then will $B' = A'''^{l-1}$, $B'' = A'''^{l-2}$, $B''' = A'''^{l-3}$, &c.; and conſequently the chance of all the (l) perſons N, M, P, &c. dying in the given time will $= 1 - B' + B'' - B''' + B'''' -$ &c.; the chance of one perſon only living during the ſame time will be $B' - 2\,B'' + 3\,B''' - 4\,B''' +$ &c; the chance of two perſons only living during the ſame time $= B'' - 3\,B''' + 4 \cdot \frac{3}{2}(6)\,B'''' - 5 \cdot \frac{4}{2} \cdot \frac{3}{3} \times B''^{5} + 6 \cdot \frac{5}{2} \cdot \frac{4}{3} \cdot \frac{3}{4}\,B''^{6} -$ &c.; the chance of three perſons only living during the ſame time $= B''' - 4\,B'''' + 5 \cdot \frac{4}{2}\,B'^{\cdot 5} - 6 \cdot \frac{5}{2} \cdot \frac{4}{3}\,B'^{\cdot 6} + 7 \cdot \frac{6}{2} \cdot \frac{5}{3} \cdot \frac{4}{4}\,B'^{\cdot 7} -$ &c.; and in general the chance of (k) perſons only living during the ſame time $= B'^{.k} - (k+1) \times B'^{.k+1} + (k+2) \,.\, \frac{k+1}{2} \times B'^{.k+2} - (k+3) \,.\, \frac{k+2}{2} \,.\, \frac{k+1}{3} \times B'^{.k+3} + (k+4) \,.\, \frac{k+3}{2} \,.\, \frac{k+2}{3} \,.\, \frac{k+1}{4} \times B'^{.k+4} -$ &c.

This cor. may either be deduced from the preceding, or eaſily inveſtigated by multiplying into each other the chances of the perſons ſuppoſed to be living or dead.

Cor. By adding the chance of all the perſons dying to the chance of only one living, there will reſult the chance of one or none living $= 1 - B'' + 2\,B''' - 3\,B'''' + 4\,B'^{.5} -$ &c.; and by adding this ſum to the chance of only two living, there will reſult the chance of two, one or none living $= 1 - B''' + 3\,B'''' - 4 \,.\, \frac{3}{2} B'^{.5} + 5 \,.\, \frac{4}{2} \,.\, \frac{3}{3}\, B'^{.6} -$ &c.; by adding this ſum to the chance of three only living, there will reſult the chance of three, two, one or none only living $= 1 - B'''' + 4\,B'^{.5} - 5 \,.\, \frac{4}{2} \times B'^{.6} + 6 \,.\, \frac{5 \cdot 4}{2 \cdot 3} \times B'^{.7} -$ &c., and in general the chance of not more than (k) perſons living to the given time $= 1 - B'^{.k+1} + (k+1)\, B'^{.k+2} - (k+2) \,.\, \frac{k+1}{2}\, B'^{.k+3} + (k+3) \,.$

$$\frac{k+2}{2}\cdot\frac{k+1}{3}B'^{\cdot k+4}-(k+4)\cdot\frac{k+3}{2}\cdot\frac{k+2}{3}\cdot\frac{k+1}{4}\times B'^{\cdot k+5}\,\&c. = 1-B'^{\cdot k+1}+(k+1)\,B'^{\cdot k+2}-\frac{k+2}{2}\,R\times B'^{\cdot k+3}+\frac{k+3}{3}\times T\times B'^{\cdot k+4}-\frac{k+4}{4}\times V\times B'^{\cdot k+5}+\&c.$$

where R, T, V, &c. denote the preceding coefficients.

This cor. can eaſily be deduced from the preceding and the ſubſequent propoſition,

$$1-n+n\cdot\frac{n-1}{2}-n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}-\ldots\pm n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\ldots\times\frac{n-l+1}{l}=\pm(n-1)\times\frac{n-2}{2}\cdot\frac{n-3}{3}\cdot\frac{n-4}{4}\ldots\frac{n-l}{l}.$$

The chance of not more than k out of l perſons living to any time will be the ſame as the chance of $(l-k)$ or more perſons dying before that time.

Ex. 7. To find the preſent value of an annuity to continue as long as (a) lives out of $(a+b)$ given ones N, M, P, Q, &c. ſhall continue in being.

From the problem it eaſily appears, that the preſent value of the annuity required for the firſt year is the aggregate of the ſeveral values of annuities for every (a) perſons contained in the abovementioned N, M, P, Q, &c. living, and the remaining (b) dying in that year, and for every ($a+1$) of thoſe perſons living and ($b-1$) dying in the ſame year; and for every $a+2$, $a+3$, $a+4$, &c. perſons living and the remaining $b-2$, $b-3$, $b-4$, &c. perſons dying in that year: and in the ſame manner may be deduced the values of the annuity for all the ſubſequent years of life.

Let S, S', S'', S''', &c. reſpectively denote the aggregates of the preſent values of all the annuities for every (a) joint lives of the abovementioned perſons N, M, P, Q, &c.; for every $a+1$, $a+2$, $a+3$, &c. joint lives of the ſame perſons N, M, P, Q, &c.; then will the aggregate of the preſent values of the annuities for every (a) of the abovementioned perſons living and the remaining (b) dying be $S-(a+1)S'+(a+1)\cdot\frac{a+2}{2}S''-(a+1)\cdot\frac{a+2}{2}\cdot\frac{a+3}{3}\times S'''+$ &c. the aggregates of the values of the annuities for every ($a+1$) of the abovementioned perſons living and ($b-1$) dying will be $S'-(a+2)S''+(a+2)\cdot\frac{a+3}{2}\times S'''-(a+2)\cdot\frac{a+3}{2}\cdot\frac{a+4}{3}S''''+$ &c.; and in the ſame manner the aggregates of the preſent values of the annuities for every $a+2$, $a+3$, $a+4$, &c. of the abovementioned perſons living and $b-2$, $b-3$, $b-4$, &c. dying will be reſpectively $S''-(a+3)S'''+(a+3)\cdot\frac{a+4}{2}S''''-$&c., $S'''-(a+4)S''''+(a+4)\cdot\frac{a+5}{2}S'''''-$&c., $S''''-(a+5)S'''''+$&c., and conſequently the preſent value of the annuity required, which is the ſum of the preceding quantities will be $S-aS'+a\cdot\frac{a+1}{2}S''-a\cdot\frac{a+1}{2}\cdot\frac{a+2}{3}S'''+a\cdot\frac{a+1}{2}\cdot\frac{a+2}{3}\cdot\frac{a+3}{4}S''''-$&c.

For the quantity $(a+1)\cdot\frac{a+2}{2}\cdot\frac{a+3}{3}\cdot\frac{a+4}{4}\cdots\frac{a+z}{z}-(a+2)\cdot\frac{a+3}{2}\cdot\frac{a+4}{3}\cdots\frac{a+z}{z-1}+(a+3)\cdot\frac{a+4}{2}\cdot\frac{a+5}{3}\cdots\frac{a+z}{z-2}-(a+4)\cdot\frac{a+5}{2}\cdot\frac{a+6}{3}\cdots\frac{a+z}{z-3}+\ldots\pm(a+z-1)\cdot\frac{a+z}{2}\mp(a+z)\pm 1=a\cdot\frac{a+1}{2}\cdot\frac{a+2}{3}\cdot\frac{a+3}{4}\cdots\frac{a+z-1}{z}$.

Ex. 8. Let an annuity of Sa at the end of the year become due, when the number of (l) perſons leſs than b, but not leſs than a, be dead; in the ſame manner let an annuity of Sb become due at the end of the year, when the number dead be leſs than c, but not leſs than b; and the ſame of Sc and Sd, when the numbers are leſs than d and e, but not leſs than c and d reſpectively, &c.; to find its preſent value.

Find the ſum of the chances of any number of perſons leſs than b, but not leſs than (a) dying in the firſt year, which multiplied into Sa, let be Ta; in the ſame manner find the ſum of the chances of any number of perſons leſs than c, but not leſs than b, dying in the ſame year, which multiplied into Sb, let be Tb; and ſimilarly of the reſt: theſe ſums can eaſily be deduced from the propoſitions before given; then will the preſent value of the expectation for the firſt year be $\frac{Ta+Tb+\&c.}{r}$: in the ſame manner find the ſum of the values of the expectation for the ſecond year, which divided by r^2 will be its preſent value; and ſimilarly may be found the ſums of the values of the expectations for the remaining years.

By the data of the problem, ſuppoſe that, of the perſons of given ages, the number of the living or dead to be given, which ſhall give title or claim to a certain annuity; and in the preceding problems, by the aid of tables of mortality, are taught the methods of finding the chance of any perſon's living or dying in every year, and conſequently the chance of the given annuity becoming due in each year, from which and the rate of intereſt can be found the preſent value of the ſaid annuity.

PROB. I.

1. To find the preſent value of an annuity (L) for a life N payable every half year, $\frac{1}{3}$ part of a year, quarterly, and in general $\frac{1}{b}$ part of a year, on the ſuppoſition of equal decrements of life between any two ſucceſſive years.

The preſent value of the annuity during the firſt year will be

$$\frac{L}{b}\times\left(\frac{(b-1)n+n'}{b\times r^{\frac{1}{b}}\times n}+\frac{(b-2)n+2n'}{b\times r^{\frac{2}{b}}\times n}+\frac{(b-3)n+3n'}{b\times r^{\frac{3}{b}}\times n}+\frac{(b-4)n+4n'}{b\times r^{\frac{4}{b}}\times n}+\ldots+\frac{n+(b-1)n'}{b\times r^{\frac{b-1}{b}}\times n}+\left(\frac{n'}{r\times n}\right)\right);$$

it being equal to $\frac{L}{b}$ $\times$ into the number of perſons living at the reſpective parts $\frac{1}{b}$, $\frac{2}{b}$, $\frac{3}{b}$, . . $\frac{b-1}{b}$, and end of the year divided by the number of perſons living at the preſent time (n) multiplied into the root of r, of which the index $\frac{1}{b}$, $\frac{2}{b}$, $\frac{3}{b}$, &c. correſponds to the part from the beginning of time, (which is the amount of 1£. at $r-1$ rate of intereſt for the part of time) and multiplied into b. In the ſame manner may be deduced the value of the

annuity ſought $\frac{L}{b \times n}\Big(\frac{(b-1)n+n'}{b \times r^{\frac{1}{b}}} + \frac{(b-2)n+2n'}{b \times r^{\frac{2}{b}}} + \frac{(b-3)n+3n'}{b \times r^{\frac{3}{b}}} + \ldots + \frac{n+(b-1)n'}{b \times r^{\frac{b-1}{b}}} + \frac{n'}{r} + \Big(\frac{(b-1)n'+n''}{br^{1\frac{1}{b}}} + \frac{(b-2)n'+2n''}{br^{1\frac{2}{b}}} + \frac{(b-3)n'+3n''}{br^{1\frac{3}{b}}} + \ldots + \frac{n'+(b-1)n''}{br^{1\frac{b-1}{b}}} + \frac{n''}{r^2}\Big) + \Big(\frac{(b-1)n''+n'''}{br^{2\frac{1}{b}}} + \frac{(b-2)n''+2n'''}{br^{2\frac{1}{b}}} + \frac{(b-3)n''+3n'''}{br^{2\frac{2}{b}}} + \ldots \frac{n''+(b-1)n'''}{br^{2\frac{b-1}{b}}} + \frac{n'''}{r3}\Big)$ + &c. to the extremity of life) $= \frac{L}{n \times b^2}\Big(\frac{b-1}{r^{\frac{1}{b}}} + \frac{b-2}{r^{\frac{2}{b}}} + \frac{b-3}{r^{\frac{3}{b}}} + \ldots + \frac{1}{r^{\frac{b-1}{b}}}\Big) \times (n + \frac{n'}{r} + \frac{n''}{r^2} + \frac{n'''}{r3}$ + &c.) $+ \frac{L}{n \times b^2}\Big(\frac{1}{r^{\frac{1}{b}}} \frac{2}{r^{\frac{2}{b}}} + \frac{3}{r^{\frac{3}{b}}} + \ldots + \frac{b-1}{r^{\frac{b-1}{b}}}\Big) \times (n' + \frac{n''}{r} + \frac{n'''}{r^2} + \frac{n''''}{r3} +$ &c.) $+ \frac{L}{n.b}\Big(\frac{n'}{r} + \frac{n''}{r^2} + \frac{n'''}{r3}$ + &c.) $= \frac{L}{b^2} \times H \times (1 + V) + \frac{L \times r}{b^2} \times K \times V + \frac{L}{b} \times V$; where $V = n' +$

$\frac{n'}{nr} + \frac{n''}{nr^2} + \frac{n'''}{nr^3} + \frac{n''''}{nr^4} +$ &c. the value of an annuity for the life of N at yearly payments;

$$H = \frac{b-1}{r^{\frac{1}{b}}} + \frac{b-2}{r^{\frac{2}{b}}} + \frac{b-3}{r^{\frac{3}{b}}} + \ldots \frac{1}{r^{\frac{b-1}{b}}} =$$

$$\frac{(b-1)\times r^{-\frac{1}{b}} - br^{-\frac{2}{b}} + r^{-1\frac{1}{b}}}{\left(1-r^{-\frac{1}{b}}\right)^2} = \frac{(b-1)r^{\frac{1}{b}} - b + r^{\frac{1-b}{b}}}{\left(r^{\frac{1}{b}}-1\right)^2};$$

$$\text{and } K = \frac{1}{r^{\frac{1}{b}}} + \frac{2}{r^{\frac{2}{b}}} + \frac{3}{r^{\frac{3}{b}}} + \ldots + \frac{b-1}{r^{\frac{b-1}{b}}} =$$

$$\frac{r^{-\frac{1}{b}} - br^{-1} + (b-1)r^{-1\frac{1}{b}}}{\left(1-r^{-\frac{1}{b}}\right)^2} = \frac{r^{\frac{1}{b}} - br^{\frac{2-b}{b}} + (b-1)r^{\frac{1-b}{b}}}{\left(r^{\frac{1}{b}}-1\right)^2}.$$

In ſome caſes it may be of a little advantage to ſubſtitute $1 + om$ for r, when om has a very ſmall proportion to r.

2. On the hypotheſis before aſſumed of equal decrements in equal times between any two ſucceſſive years, and the preſent value of annuities at yearly payments for the joint lives of (k) perſons of N, M, P, Q, &c. years of age, of which the complements are as before n, m, p, q, &c.; and alſo the preſent values of yearly annuities for any (k) joint lives, of which ſome are of $N-1$, $M-1$, $P-1$, $Q-1$, &c. years of age, and the remaining ones of N, M, P, Q, &c.; but

with this exception, that the lives of N and $N-1$, M and $M-1$, P and $P-1$, &c. reſpectively are never joined together in the abovementioned (k) lives, being given; the preſent value of the abovementioned (k) joint lives of N, M, P, Q, &c. years of age at half yearly, $\frac{1}{3}$ part, quarterly, and generally $\frac{1}{b}$ part of a year payments can be deduced:

From the preceding principles it appears that the preſent value of the annuity beforementioned will be equal to $\frac{L}{b \times n \times m \times p \times q \times \&c.}$

$$\times \left(\frac{(b-1)n+n'}{b \times r^{\frac{1}{b}}} \times \frac{(b-1)m+m'}{b} \times \frac{(b-1)p+p'}{b} \times \frac{(b-1)q+q'}{b} \times \&c. + \frac{(b-2)n+2n'}{b \times r^{\frac{2}{b}}} \times \frac{(b-2)m+2m'}{b} \right.$$

$$\frac{(b-2)p+2p'}{b} \times \frac{(b-2)q+2q'}{b} \times \&c. + \frac{(b-3)n+3n'}{b\,r^{\frac{3}{b}}}$$

$$\times \frac{(b-3)m+3m'}{b} \times \frac{(b-3)p+3p'}{b} \times \&c. + \ldots\ldots +$$

$$\frac{n+(b-1)n'}{b \times r^{\frac{b-1}{b}}} \times \frac{m+(b-1)m'}{b} \times \frac{p+(b-1)p'}{b} \times \&c.$$

$$\left. + \left(\frac{n'm'p'q'\&c.}{r} \right) \right) + \frac{(b-1)n'+n''}{b\,r^{1\frac{1}{b}}} \times \frac{(b-1)m'+m''}{b}$$

$$\times \frac{(b-1)p'+p''}{b} \times \frac{(b-1)q'+q''}{b} \times \&c. +$$

(b—2)

$$\frac{(b-2)n'+2n''}{br^{1\frac{2}{b}}}\times\frac{(b-2)m'+2m''}{b}\times\frac{(b-2)p'+2p''}{b}\times$$

$$\&c.+\ldots+\frac{n'+(b-1)n''}{br^{1\frac{b-1}{b}}}\times\frac{m'+(b-1)m''}{b}\times$$

$$\frac{p'+(b-1)p''}{b}\times\&c.+\&c.+\left(\frac{n''m''p''q''\&c.}{r^2}\right)+\&c)$$

all the letters denoting the ſame quantities as before.

It is evident from multiplying theſe factors into each other, that the contents reſulting are eaſily reducible into ſerieſes before given for finding the values of annuities.

On the ſame hypotheſis, and from the principles taught in theſe two caſes, can be deduced the preſent values of the annuities contained in all the former propoſitions by half yearly, $\frac{1}{3}$ part of the year, quarterly, and in general $\frac{1}{b}$ part of the year payments.

PROB. II.

1. From the preſent value of an annuity (*L*) for a life of *N* years of age; to find the ſame value for a life of *N*—*e* years of age.

Let $n_{\prime}, n_{\prime\prime}, n_{\prime\prime\prime}, \ldots\ n_{e-2}, n_{e-1}, n_{e}$ reſpectively denote the number of perſons living contained in the given tables of mortality at the ages of *N*—1, *N*—2, *N*—3, . . . *N*—*e*+2, *N*—*e*+1, *N*—*e*;

then will the preſent value of a life of *N*—*e* years of age be

$$L\times\left(\frac{\underset{e-1}{n}}{\underset{e}{n}\times r}+\frac{\underset{e-2}{n}}{\underset{e}{n}\,r^{2}}+\frac{\underset{e-3}{n}}{\underset{e}{n}\,r^{3}}+\ldots\frac{\underset{'''}{n}}{\underset{e}{n}\,r^{e-3}}+\frac{\underset{''}{n}}{\underset{e}{n}\,r^{e-2}}+\frac{\underset{'}{n}}{\underset{e}{n}\,r^{e-1}}+\frac{n}{\underset{e}{n}\,r^{e}}\right)+\frac{n\times V}{\underset{e}{n}\,r^{e}};$$

where *V* denotes the preſent value for a life of *N* years of age.

2. From the preſent value (*W*) of an annuity *L* for the joint lives of perſons of *N*, *M*, *P*, *Q*, &c. years of age being given, to find the preſent value of the annuity (*L*) for the joint lives of perſons of *N*—*e*, *M*—*e*, *P*—*e*, *Q*—*e*, &c. years of age.

Let a ſimilar notation to the preceding given for the number of perſons living at the ages of *N*—1, *N*—2, . . . , *N*—*e* be applied to denote the number of perſons living at the ages *M*—1, *M*—2, . . ., *M*—*e*; *P*—1, *P*—2, . . ., *P*—*e*; *Q*—1, *Q*—2, . . . , *Q*—*e*; &c. the value required will be

$$L\times\left(\frac{\underset{e-1}{n}}{\underset{e}{n}\times r}\times\frac{\underset{e-1}{m}}{\underset{e}{m}}\times\frac{\underset{e-1}{p}}{\underset{e}{p}}\times\frac{\underset{e-1}{q}}{\underset{e}{q}}\times\text{\&c.}+\frac{\underset{e-2}{n}}{\underset{e}{n}\times r^{2}}\times\frac{\underset{e-2}{m}}{\underset{e}{m}}\times\frac{\underset{e-2}{p}}{\underset{e}{p}}\times\frac{\underset{e-2}{q}}{\underset{e}{q}}\times\text{\&c.}+\ldots+\frac{\underset{1}{n}}{\underset{e}{n}\,r^{e-1}}\times\right.$$

m ×

$$\frac{\overset{m}{1}}{\underset{e}{m}} \times \frac{\overset{p}{1}}{\underset{e}{p}} \times \frac{\overset{q}{1}}{\underset{e}{q}} \times \text{\&c.} + \frac{n}{\underset{e}{n}\, r^{e}} \times \frac{m}{\underset{e}{m}} \times \frac{p}{\underset{e}{p}} \times \frac{q}{\underset{e}{q}} \times \text{\&c.}) +$$

$$\frac{n \,.\, m \,.\, p \,.\, q \,.\, \text{\&c.}}{\underset{e}{n} \,.\, \underset{e}{m} \,.\, \underset{e}{p} \,.\, \underset{e}{q} \,.\, \text{\&c.}\; r^{e}} \times W.$$

3. The preſent value of an annuity dependent on the living of (*a*) perſons and dying of *b* perſons of given ages, can be deduced from the values of annuities for the abovementioned (*a*) joint lives, and alſo for the ſame lives joined with the *b*, *b*—1, *b*—2, *b*—3, . . . , 2, 1, 0 lives: from the ſame principles and data may be deduced the preſent value of an annuity dependent on the living of (*a*) perſons, and dying of (*b*) perſons, of which the reſpective ages are 1, 2, 3, &c. years younger or older than the preceding.

4. From the preſent values of annuities for every *a* and *a*—1 of the beforementioned lives joined with the *b*, *b*—1, *b*—2, . . . , 2, 1, 0 lives, may be deduced the preſent value of this annuity payable at half yearly, quarterly and every $\frac{1}{b}$ part of a year payments.

5. From this annuity by the preceding principles may be found the ſum of a ſimilar annuity for perſons, of whom each is 1, 2, 3, &c. years

younger than the perſons N, M, P, Q, &c.: for by the preceding propoſition from the aggregates S, S', S'', S''', &c. before deduced, may be found ſimilar aggregates for perſons, of which the ages are 1, 2, 3, &c. years leſs than the ages of the beforementioned perſons.

The principles before given may be applied to find the preſent value of the annuity at half yearly, quarterly, &c. payments.

PROB. III.

1. If a perſon is entitled to a ſum of money (A) on the contingency of a certain event, and to an equal ſum of of money (A) on the contingency of a ſimilar event after the happening of the firſt, and ſo on repeatedly for ever; to find the preſent value of all the ſucceſſive expectations. Let the preſent value of the firſt expectation be B, then will the preſent value of all the ſucceſſive expectations be $B + \frac{B^2}{A} + \frac{B^3}{A^2} + \frac{B^4}{A^3} + \text{\&c.} = \frac{AB}{A-1}$.

2. Let the preſent value of the firſt expectation A be B, of the ſecond C at the time of receiving the firſt be D; of the third E at the time of receiving the ſecond be F, &c.; then

will the preſent values of the ſucceſſive expectations be B, $\frac{BD}{A}$, $\frac{B \times D \times F}{A \, . \, C}$, &c.

1. When the preſent value of a ſum of money to be received immediately on the contingency of events be required, of which the probability of happening is known, then the expectation, probability, &c. depends not only on the contingency of events before the end or beginning of a year or time, of which the probability is contained in the tables, as annuities; but on the intermediate time contained between any two ſucceſſive ones, of which the probabilities are contained in the given tables.

In this caſe ſome hypotheſis muſt be aſſumed for the intermediate times, and perhaps none or very few will be ſo eaſy, or anſwer more agreeably to the truth, than the hypotheſis during thoſe little intervals (particularly in middle age) of equal decrements in equal times.

2. If the ſum of money (S) or advantage, &c. is to be paid immediately and not at ſtated times on certain events happening; the value of the preſent expectation of the events happening at a given time is equal to the ſum of all the chances of the events happening at that time × S; diſcount being allowed from the preſent to the time at $r-1$ rate of intereſt.

3. When regard is to be had to the order, in

which the events ſucceed each other; the chance of one ſucceeding another may be different as eſtimated from different hypotheſes, but if in calculating the values of annuities, &c. the hypotheſis of equal decrements in equal times be aſſumed, and the ſame limit is aſſigned to their continuance in life, or in other words they have the ſame complements of life; then will the chance of the former dying before the latter be equal to the chance of the latter dying before the former; and forther if α be the number of ſuch perſons, required to die in a given order; then the probability of their dying in the order mentioned, to their not dying in that order, will be as $1 : 1.2.3.4\ldots(\alpha-1).\alpha-1$.

4. To find the probability of ſurvivorſhip between two perſons *A* and *B*, whoſe ages are *N* and *M*. Let n and m denote the number of perſons in the table living at the ages *N* and *M* as before, and H and H' the number of perſons living at the ages $L+N$ and $L+N+1$, and I and I' the number of perſons living at the ages $L+M$ and $L+M+1$, and let x be any diſtance of time from $L+N$ contained between $L+N$ and $L+N+1$; then by the hypotheſis of equal decrements; 1 year $: x :: H-H'$: the decrements of life in the time x; and conſequently $H-x\times(H-H')$ may be ſuppoſed the number of perſons living at the time x or age $L+N+x$; and

$\frac{H-x\times(H-H')}{n}$ the chance of a perſon at the age N living to the age $L+N+x$; from the ſame principles the preſent chance of a perſon of the age $L+M$ dying at the time $\dot{x}$ will be $(I-I')\times\frac{\dot{x}}{m}$; and the probability of A's ſurviving B at the time $\dot{x}$ will be as the chance of B's dying at that time and A's living to it, and conſequently as $(I-I')\times\frac{H-x\times(H-H')}{n\times m}\times\dot{x}$, of which the fluent contained between the values 0 and 1 of x, will reſult $(I-I')\times\frac{H-\frac{1}{2}(H-H')}{n\times m}=(I-I')\times\frac{(H+H')}{2n\times m}$: which is the probability of ſurvivorſhip of A between the ages $N+L$ and $N+L+1$ of A, and $M+L$ and $M+L+1$ of B, and ſo of every other year during their continuance in life.

2. If A is to receive a ſum of money (S), provided he ſurvives B, then the chance of his becoming entitled to it between the ages $N+L$ and $N+L+1$ will be, as in the preceding caſe, $(I-I')\times\frac{H+H'}{2\,n\,m}$, and the preſent value of it, if it is to be paid not before the year of his age $N+L+1$ will be that quantity $\times\ S$ and divided by r^{L+1}, where r the amount of 1£. for a year as before; but if it is to be paid immediately, then find the fluent of the quantity $S\times(I-I')\times$

H—

$$\frac{H-x\times(H-H')}{n\,.\,m\,.\,r^{L+x}}\times\dot{x}=S\times(I-I')\times\frac{a+bx}{n\,.\,m\,.\,r^{x}}\times\frac{1}{r^{L}}$$

contained between the ſame values 0 and 1 of x, which will be the preſent value of the required expectation, where $b=\frac{H-H'}{\log.\,r}$ and $a=\frac{b-H}{\log.\,r}$; and in the ſame manner find the expectation for every other year during life, which being added together gives the whole expectation. The fluents of the fluxions $r^{-x}\times\dot{x}$, $r^{-x}\times x\dot{x}$, and in general $r^{-x}\times x^{m}\dot{x}$, m being a whole number, are known.

5. To find the preſent value of a given ſum of money S to be received if one man A, whoſe age is N, ſurvives any number (α) of perſons B, C, D, E, &c. and of which B ſurvives C, C ſurvives D, &c.: from the preceding principles find the chance of A, whoſe age is N, living to the age $N+L+x$, and alſo the chance of B dying in the time $\dot{x}$; and further the chances of C, D, &c. dying before they have lived $L+x$ years; all of which being multiplied into each other, and divided by $1\,.\,2\,.\,3\,..\,(\alpha-1)$, ſuppoſe $V\times\dot{x}$; the fluent, corrected as before, of the reſulting fluxion and divided by r^{L+1}, if the ſum is not to be paid before the end of the year, will give the preſent value of the expectation in the year contained between $N+L$ and $N+L+1$ of his age: but if the ſum is to be received immedi-

ately on its becoming due, find the fluent of $\frac{V\dot{x}}{r^x} \times \frac{1}{r^L}$ contained between the beforementioned values 0 and 1 of x, and it will give the preſent value of the chance of 1£. for that year, and in the ſame manner may be found the values of the chances for every year, of which the ſum multiplied into S gives the preſent value of the expectation required.

6. If the order of ſurvivorſhip for the perſons B, C, D, &c. is not given, then find the abovementioned expectation when B is the ſurvivor, and dies at $\dot{x}$, and in the ſame manner when C, D, &c. are reſpectively the ſurvivors, and die at $\dot{x}$; and the aggregate of all theſe expectations let be $W\dot{x}$, the fluent of which fluxion contained between the values 0 and 1 of x divided by r^{L+1}; or the fluent of the fluxion of $\frac{W\dot{x}}{r^x} \times \frac{1}{r^L}$ contained between the ſaid value of x will give the value of the expectation of (*A)* between the $N+L$ and $N+L+1$ years of his age: and in the ſame manner may be deduced his expectation in every other year, and thence the ſum of them or the whole expectation.

7. If the expectation depends on the chance of b perſons dying out of k perſons, whoſe ages are given, and *A*'s; whoſe age is likewiſe given, ſurviving; find by the preceding principles the

chance of any one of the (b) perſons dying at $\dot{x}$, and the chances of any b—1 of the remaining ones dying before that time $L+x$, and the chances of the remaining k—b perſons and A living to the ſaid time $L+x$; and ſo of each of them; the ſum of all theſe chances ſuppoſe $V\dot{x}$, find the fluent of $\frac{S \times V\dot{x}}{r^{L+1}}$ or $\frac{S \times V\dot{x}}{r^{L+x}}$, as before; and it will be the preſent value of the expectation for the year abovementioned.

In the ſame manner may ſimilar queſtions relating to the chances of perſons living or dying, whoſe ages are given, be anſwered.

Cor. As the ſum or whole value of all theſe expectations are reducible into the values of the joint lives of the given perſons, multiplied by given quantities, and as the values of all ſuch joint lives, &c. have been before ſhewn to be deducible from the value of the joint lives, &c. of perſons one, two, three, &c. years older than them, it follows that theſe expectations may be reduced from ſimilar expectations of perſons one, two, three, &c. years younger than them; and vice verſâ the expectations of perſons younger may be deduced from thoſe of the older.

Life being very uncertain, and liable to an almoſt infinite variety of accidents, which arreſt it in its career towards old age, cannot be ſubject to human calculations, but will vary principally from them in ages which are moſt infirm, and conſequently ſubject to the greateſt number of accidents deſtructive of it, as in infancy and old age: tables of mortality in thoſe periods will generally moſt differ from each other; but in old age, where the ſurvivors are very few, the tables will generally be very irregular; in older years it will ſometimes happen, that the numbers are greater than in younger, the ſame may alſo happen to the increments in younger life; theſe irregularities have probably been corrected by their publiſhers.

Calculations of the preſent values of lives from different real tables of mortality, cannot be ſuppoſed to agree to a very great degree of accuracy: in ſelling an annuity, &c. the conſtitution of the purchaſer ſhould be conſidered, as ſeldom any perſon of an infirm ſtate of health will purchaſc an annuity, therefore ſomething more than the value, as calculated from bills of mortality, ſhould be paid; as in the tables are included perſons of all conſtitutions and ſtates of health. This is one reaſon that tables of tontine in ſuch caſes ſhould be preferred, as the ſubſcribers, at the time of purchaſe, may generally be ſuppoſed to enjoy a good ſtate of health.

The decrements of life half yearly, quarterly, or for any period leſs than a year, may be aſſumed proportional to the time with as great a degree of accuracy as the ſubject will admit, unleſs they border on infancy or old age; the ſame propoſition may, in many caſes, be extended to much longer periods of time.

If the expectation of life could be ſuppoſed a conſtant algebraic function Q of x the age, and in it be ſubſtituted for x ſucceſſively a, $a+d'$, $a+d''$, $a+d'''$, &c.; and the quantities reſulting be A, B, C, D, &c. if d', d'', d''', &c. bear a very ſmall ratio to the diſtance of (a) from the values of x, when $Q=o$, then will $A-B$, $A-C$, $A-D$, &c. be nearly in the ſame ratio as d', d'', d''',&c.

The function Q muſt become $=0$, when life in general ends, i. e. at the complement of life.

Let the expectation of life be denoted by $\frac{N-x}{N}$, where N denotes the complement of life either in years, half years, &c. i. e. the diſtance of the preſent from extreme age, and x the age in the ſame: when it denotes years, it becomes the hypotheſis of Mr. D'Moivre.

I do not ſay, that the values of annuities, &c. calculated from this hypotheſis will *very* nearly agree to the ſame values calculated from tables, the very learned and accurate Baron Maſeres has proved, that they will not.

1. The value of an annuity L for a life, whoſe complement is N at $r-1$ rate of intereſt, (i. e. the intereſt of 1£. for a year, half year, &c.;

according as N and x denote years, half years, &c.); will be

$$\frac{L}{N}\left(\frac{N-1}{r}+\frac{N-2}{r^2}+\frac{N-3}{r^3}\ldots\frac{N-N=0}{r^N}\right)=\frac{L}{N}\left(\frac{r^N-1}{(r-1).r^N}\times N-\frac{r^{N+1}-(N+1)r+N}{(r-1)^2\times r^N}\right)=\frac{L}{N}\times\frac{(N-1)\times r^{N+1}-Nr^N+r}{(r-1)^2.r^N}.$$

2. The value of an annuity L during the continuance of any one of two lives, whoſe complements are N and M reſpectively (of which N is equal or greater than M) at the ſame $r-1$ rate of intereſt will be

$$\frac{L}{N\times M}\left(\frac{NM-1}{r}+\frac{NM-4}{r^2}+\frac{NM-9}{r^3}+\ldots\frac{NM-M^2}{r^M}+\frac{L}{N}\left(\frac{N-M-1}{r^{M+1}}+\frac{N-M-2}{r^{M+2}}+\frac{N-M-3}{r^{M+3}}+\ldots\frac{N-N=0}{r^N}\right).\right.$$

The chance of each N and M dying in the firſt year will be reſpectively $\frac{1}{N}$ and $\frac{1}{M}$, and conſequently both dying in the firſt year $\frac{1}{N}\times\frac{1}{M}$; the chance of both or one of the two living the firſt year $\left(\frac{N-1}{N}\times\frac{M-1}{M}+\frac{N-1}{N}\times\frac{1}{N}+\frac{M-1}{M}\times\frac{1}{N}=\frac{NM-1}{NM}\right)$ is the chance of both dying deducted from certainty, and conſequently $= 1 -$

$-\frac{1}{N\times M}=\frac{NM-1}{N\times M}$, and the value of its preſent expectation $\frac{NM-1}{N\times M\times r}$; by the ſame mode of reaſoning, the chance of both or one of the two perſons living two years is $\frac{NM-4}{N\times M}$; and ſo on; whence the preſent value of an annuity L for the longeſt of the above two lives is $\frac{L}{N\times M}$ $\left(\frac{NM-1}{r}+\frac{NM-4}{r^2}+\frac{MN-9}{r^3}+\dots+\frac{NM-M^2}{r^M}\right)$ + the value of the chance of the youngeſt life after the certainty of the death of eldeſt $=\frac{L}{N}\left(\frac{N-M-1}{r^{M+1}}+\frac{N-M-2}{r^{M+2}}+\frac{N-M-3}{r^{M+3}}\dots+\frac{N-N=0}{r^N}\right)$.

Ex. 3. Let it be required to find the preſent value of an annuity L at $r-1$ rate of intereſt for the life of the longeſt liver of any number (b) of perſons, whoſe complements of life are reſpectively N, M, R, S, T, V, &c.; N being leſs than M, M than R, R than S, &c.

The preſent value of the expectation will be $\frac{L}{N\times M\times R\times S\times \&c.=A}\left(\frac{A-1}{r}+\frac{A-2^b}{r^2}+\right.$

$A-$

$$\frac{A-3^b}{r^3}+\ldots\frac{A-N^b}{r^N}\Big)+\frac{L}{A}\Big(\frac{A-(N+1)^{b-1}\times N}{r^{N+1}}$$

$$+\frac{A-(N+2)^{b-2}\times N}{r^{N+2}}+\frac{A-(N+3)^{b-1}\times N}{r_N{}^{+3}}+$$

$$\ldots+\frac{A-M^{b-1}\times N}{r^M}+\frac{L}{A}\Big(\frac{A-(M+1)^{b-2}\times N\times M}{r^{M+1}}$$

$$+\frac{A-(M+2)^{b-2}\times N\times M}{r^{M+2}}+\frac{A-(M+3)^{b-2}\times N\times M}{r^{M+3}}$$

$$+\ldots+\frac{A-R^{b-2}\times N\times M}{r^k}\Big)+\frac{L}{A}\Big(\frac{A-(R+1)^{b-3}\times N\times M\times R}{r^{R+1}}$$

$$+\frac{A-(R+2)^{b-3}\times N\times M\times R}{r^{R+2}}+\ldots$$

$$\frac{A-S^{b-3}\times N\times M\times R}{r^S}\Big)+\&c.$$

The ſum of each of theſe ſeries can eaſily be deduced from the lemmas before given, and conſequently the preſent value of the expectation required.

The chances of the living or dying of any perſons of given ages, being given by the hypotheſis, and the ſeveral expectations in conſequence of their living or dying at every different period being deduced from the data and the preceding principles, the aggregate required of all the expectations at the ſeveral different periods may be found.

From the preceding principles algebraical quantities may be tranſlated into expectations from lives, deaths, &c.

In this caſe, ſimilar to the preceding, the ſame quantity, contained in different formulæ, may be tranſlated into the aggregates of the values of different expectations, which conſequently are equal to each other.

CORRIGENDA.

Pag. 3. lin. 2. *for* Medit. Anal. *read* Phil. Tranſ.; for b', b'', b''', &c. read b; pag. 6. l. 13. *dele* of A's happening n times; l. 16. dele $n+$; pag. 7. l. 3. *dele* $\times\frac{a^n}{N^n}$; pag. 9. l. 23. *for* if, *read* if in; l. 29. for Na', read a'; l. ult. *for* equation, *read* equations; and for $N\times a'$, read a'; pag. 12. *for* ſum of the chances, &c., *read* ſum of the chances of the events A and B happening n times in n trials; l. penult. *for* once, *read* once multiplied; pag. 14. l. 18. *for* k^{th}, *read* every k^{th}.